EXAMPLES OF PROPER, CLOSED, WEAKLY DENSE SUBSPACES IN NON LOCALLY CONVEX F-SPACES*

BY

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ABSTRACT

It is shown that any F-space having l^p as a continuous linear image for some $0 < p < 1$ must contain proper closed subspaces which are dense in the weak topolgy. In particular, when $0 < p < 1$, such subspaces are present in l^p, and the same is shown to hold in certain F-spaces of analytic functions which nevertheless have enough continuous linear functionals to separate points.

1. **Introduction.** We give some examples of F-spaces (complete linear metric spaces) which, despite having a separating family of continuous linear functionals contain proper, closed subspaces which are dense in the weak topology. It follows from the Hahn Banach theorem that these spaces are not locally convex; but it is not known if *every* non-locally convex F-space must contain a proper, closed, weakly dense (PCWD) subspace (see [4, sec. 7]).

This phenomenon seems first to have occurred in the literature in the work of Duren, Romberg and Shields [4], who discovered that the Hardy spaces H^p $(0 < p < 1)$, which have separating dual spaces, contain PCWD subspaces. Earlier Peck [9] found closed subspaces in l^p $(0 < p < 1)$ which were not weakly closed, however his subspaces were not weakly dense.

In Section 2 of this paper we obtain some results about F-spaces which show, in particular, that l^p contains PCWD subspaces when $0 < p < 1$. In Sections 3 and 4 we apply our results to certain spaces of analytic functions. Specifically we consider the spaces $E(p,\tau)$ $(0 < p < 1,\ 0 < \tau \infty <)$ of entire functions f of exponential type τ with

$$\|f\| = \int_{-\infty}^{\infty} |f(x)|^p \, dx < \infty, \tag{1.1}$$

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and the spaces A^p $(0 < p < 1)$ of functions f analytic in the open unit disc with

$$\|f\| = \iint_{|z|<1} |f(z)|^p dxdy < \infty. \tag{1.2}$$

We show that these spaces are complete, have separating families of continuous linear functionals, and contain PCWD subspaces. As a by-product of our method we are able to determine the modulus of convexity (see Section 2) of each of these spaces. Our method also applies to other spaces, including H^p $(0 < p < 1)$, where it produces PCWD subspaces in a manner quite different from that of [4], and provides a different proof of Landsberg's theorem that H^p has modulus of convexity p [6, Satz 3].

2. Results on F-spaces, with application to l^p $(0 < p < 1)$.

In this section we show that any F-space which can be mapped onto l^p $(0 < p < 1)$ by a continuous linear transformation contains PCWD subspacs and has modulus of convexity at most p. In particular when $0 < p < 1$, l^p has PCWD subspaces.

Recall that if $0 < p \leqq 1$, a *p-norm* (*norm* if $p = 1$) on a linear space is a non-negative, subadditive functional $\|\cdot\|$ vanishing only at the origin such that

$$\|\alpha x\| = |\alpha|^p \|x\|$$

for each vector x and scalar α. A *p-normed space* is a linear topological space whose topology is induced by a p-norm. A subset S of a linear space is called *p-convex* (*convex* if $p = 1$) if $\alpha x + \beta y \in S$ whenever $x, y \in S$ and α, β are non-negative numbers with $\alpha^p + \beta^p = 1$. Note that every p-convex set is r-convex for $0 < r \leqq p$, and that the unit ball of a p-normed space is p-convex. If E is a linear topological space, the supremum of the set of p for which E has a local base of p-convex sets will be called the *modulus of convexity* of E, and denoted by $k(E)$. This supremum need not be attained; for example there are non-locally convex spaces with modulus of convexity 1 (see [10] for an example). Every p-normed space has modulus of convexity $\geqq p$.

We begin with a well known basic result.

PROPOSITION 1. *An F-space contains PCWD subspaces if and only if it has as a continuous linear image an infinite dimensional F-space with trivial dual (i.e. no nontrivial continuous linear functional).*

Proof. Suppose E is an F space, and K is a PCWD subspace. Then E/K is an F-space [5, p. 167] which is a continuous linear image of E under the quotient map. We claim that K has infinite codimension in E. Suppose not. Let H be any algebraic complement of K in E. Then H is a closed, finite dimensional subspace, [5, pp. 159–160] whose relative topology is that of Euclidean space. Thus H supports a nontrivial continuous linear functional, which may be extended to all of E by defining it to be 0 on K. There is, then, a continuous nontrivial linear functional on E which vanishes on K; contradicting the fact that K is weakly dense. Thus K has infinite codimension, so E/K is infinite dimensional.

Suppose μ is a nontrivial continuous linear functional on E/K. Then the equation

$$\lambda(f) = \mu(f+K) \qquad (f \text{ in } E) \tag{2.1}$$

defines a continuous nontrivial linear functional λ on E which vanishes on K, contrary to the fact that K is weakly dense. Thus E/K can have no nontrivial continuous linear functional.

Conversely, suppose E and F are F-spaces, and F is infinite dimensional with trivial dual. Let T be a continuous linear map taking E onto F. Clearly $K = \ker T$ is a proper, closed subspace of E. We claim it is weakly dense. If not, then there is a nontrivial continuous linear functional λ on E which vanishes on K.

Equation (2.1) then defines a nontrivial continuous linear functional μ on E/K. However the equation

$$U(f+K) = Tf \qquad (f \text{ in } E)$$

defines an algebraic isomorphism of E/K onto F. If $Q: E \to E/K$ denotes the quotient map, then $U^\circ\, Q = T$ is continuous, hence U is continuous. It follows from the interior mapping principle [5, p. 170] that U is a homeomorphism. Thus E/K is linearly homeomorphic with F, and can therefore have no nontrivial continuous linear functional. This is a contradiction, therefore K must be weakly dense. ///

In particular we can show that $l^p (0 < p < 1)$ contains a PCWD subspace by mapping it onto $L^p = L^p([0,1])$, which has a trivial dual (see [5, p. 162], for example). The existence of such a mapping follows from the next theorem, which is well known for $p = 1$ [5, p. 283].

PROPOSITION 2. *Every complete, separable p-normed space* $(0 < p \leqq 1)$ *is a continuous linear image of* l^p.

Proof. Let $S = \{e_n: n \geqq 0\}$ be a countable dense subset of $U = \{f \text{ in } E: \|f\| \leqq 1\}$. If $x = (\xi_j)$ belongs to l^p, and $n, k \geqq 0$, then

$$\left\| \sum_{j=n}^{n+k} \xi_j e_j \right\| \leqq \sum_{j=n}^{n+k} |\xi_j|^p.$$

The sequence $(\sum_0^n \xi_j w_j: n \geqq 0)$ is therefore Cauchy in E, so it converges to an element $\sum_0^\infty \xi_j e_j$ of E. Define the map $T: l^p \to E$ by

$$Tx = \sum_0^\infty \xi_j e_j \qquad (x = (\xi_j) \text{ in } l^p).$$

If x belongs to l^p, then $\|Tx\| \leqq \|x\|$, so T is a continuous linear map. We show that $T(l^p) = E$. Suppose f is in E and $\|f\| \leqq 1$. Since S is dense in U, there exists an index $n(0)$ such that

$$\|f - e_{n(0)}\| \leqq 2^{-p}.$$

Thus $f - e_{n(0)}$ belongs to $\frac{1}{2}U$, which contains $\frac{1}{2}S$ as a dense subset. Consequently there exists an index $n(1)$ such that

$$\|f - e_{n(0)} - \tfrac{1}{2}e_{n(1)}\| \leqq 4^{-p}.$$

Continuing in this manner we obtain a sequence $n(k)$ $(k = 0,1,2,\cdots)$ of indices such that

$$\left\| f - \sum_{j=0}^{k} 2^{-j} e_{n(j)} \right\| \leqq 2^{-(k+1)p}$$

$(k = 0,1,2,\cdots)$. Let x be the sequence having 2^{-k} in position $n(k)$, and 0 elsewhere. Then x belongs to l^p, and $Tx = f$. ///

COROLLARY 1. *Any F-space having l^p $(0 < p < 1)$ as a continuous linear image contains proper, closed, weakly dense subspaces. In particular, l^p itself contains such subspaces.*

Proof. If follows from Proposition 2 that L^p is a continuous linear image of l^p. If there is a continuous linear map taking the F-space E onto l^p, then there is one taking E onto L^p, and the result follows from Proposition 1. ///

It is well known that $k(L^p) = p$ if $0 < p \leqq 1$. The next proposition provides another proof of this, along with the fact that L^p $(0 < p < 1)$ has no nontrivial continuous linear functional.

PROPOSITION 3 (cf. [2, V.7.37]). *If $p < r \leqq 1$, then the r-convex hull of the L^p unit ball is all of L^p.*

Proof. The r-convex hull of the L^p unit ball is the set of elements $\sum_1^n \alpha_i x_i$, where $\| x_i \| \leqq 1$, $\alpha_i \geqq 0$ and $\sum_1^n \alpha_i^r = 1$ $(n = 1,2,\cdots)$. Suppose f belongs to L^p. Choose $n > 0$ such that $\| f \|^{p/r-1} \leqq 1$, and partition the unit interval into disjoint subintervals $I(1), I(2), \cdots, I(n)$, such that

$$\int_{I(n)} |f|^p dx = \| f \| / n.$$

Let f_j coincide with $n^{1/r} f$ on $I(j)$, and vanish on the rest of the unit interval $(j = 1,2,\cdots,n)$. Then $f = n^{-1/r}(f_1 + f_2 + \cdots + f_n)$, and $\| f_j \| = n^{p/r-1} \| f \| \leqq 1$ $(j = 1,2,\cdots,n)$. ///

Corollary 1. L^p $(0 < p < 1)$ *has no nontrivial continuous linear functional.*

Proof. Every continuous linear functional on L^p is bounded on the unit ball, hence on the convex hull of the unit ball, which is the whole space. Therefore the functional vanishes identically. ///

Corollary 2. *If* $0 < p < 1$, *then* $k(L^p) = p$.

Proof. For $p < r \leqq 1$, it follows from Proposition 3 that the only nonempty r-convex open set is L^p itself. Thus $k(L) \leqq p$. On the other hand, every p-normed space has modulus of convexity $\geqq p$; and the result follows. ///

Proposition 4. *Let E and F be F-spaces. If F is a continuous linear image of E, then* $k(E) \leqq k(F)$.

Proof. Suppose T is a continuous linear map of E onto F. Suppose $k(F) < r \leqq 1$. It is enough to show that E does not have a local base of r-convex sets. If $\{U\}$ is such a base, then it follows from the continuity of T and the interior mapping principle that $\{T(U)\}$ is a local base in F consisting of r-convex sets. Thus $k(F) > p$, contradicting our original assumption. ///

Corollary. *For* $0 < p < 1$, *the modulus of convexity of* l^p *is* p.

In the following theorem, most of which has already been proven, we summarize those results which will be needed later on.

Theorem 1. *Suppose E is an F-space having l^p as a continuous linear image, for some $0 < p < 1$. Then E has proper, closed, weakly dense subspaces, and $k(E) \leqq p$. If E is p-normed, then $k(E) = p$.*

Proof. That E has PCWD subspaces is Corollary 1 of Proposition 2. That

$k(E) \leqq p$ follows from Proposition 4 and its corollary. If E is p-normed, then $k(E) \geqq p$. ///

3. The spaces $E(p,\tau)$ $(0 < p < 1,\ 0 < \tau < \infty)$.

Recall that $E(p,\tau)$ denotes the space of entire functions of exponential type τ whose restrictions to the real axis belong to $L^p(-\infty,\infty)$. More specifically, f belongs to $E(p,\tau)$ if and only if for each $\varepsilon > 0$,

$$\max_{|z|=r} |f(z)| = 0(\exp(\tau+\varepsilon)r) \quad \text{as } r \to \infty, \tag{3.1}$$

and

$$\| f \|_p^p = \int_{-\infty}^{\infty} |f(x)|^p dx < \infty. \tag{3.2}$$

If $1 \leqq p < \infty$, then $\|f\| = \|f\|_p$ is a norm; while if $0 < p < 1$, then $\|f\| = \|f\|_p^p$ is a p-norm. It is well known (see Proposition 5 and its proof) that $E(p,\tau)$ $(0 < p < \infty)$ can be considered as a closed subspace of $L^p(-\infty,\infty)$ and (3.1) can be replaced by

$$\max_{|z|=r} |f(z)| = 0(\exp(\tau r)) \quad \text{as } r \to \infty. \tag{3.3}$$

In particular the spaces $E(p,\tau)$ are complete. In this section we show that when $0 < p < 1$ they have separating families of continuous linear functionals, contain PCWD subspaces, and have modulus of convexity p.

The complex function theory that we use can be found, for the most part, in Boas [1, sec. 6.7]. Of crucial importance is the following inequality of Plancherel and Pólya [1, p. 98].

THEOREM A. *If f belongs to $E(p,\tau)$ $(0 < p < \infty)$, then*

$$\int_{-\infty}^{\infty} |f(x+iy)|^p dx \leqq \|f\|_p^p \exp(p\tau|y|).$$

PROPOSITION 5. *The spaces $E(p,\tau)$ $(0 < p < \infty)$ are complete, and the evaluation functionals $\{\lambda_z : z \text{ complex}\}$ defined by*

$$\lambda_z(f) = f(z)$$

form a separating family of continuous linear functionals.

Proof. If f belongs to $E(p,\tau)$, then $|f|^p$ is subharmonic, so for each z,

$$|f(z)|^p < \pi^{-1} \iint_{|z-w|<1} |f(w)|^p du dv$$

$$\leqq \pi^{-1} \int_{y-1}^{y+1} \int_{-\infty}^{\infty} |f(u+iv)|^p du dv.$$

Using Theorem A to estimate the inner integral, we obtain

$$|f(z)|^p \leqq \pi^{-1} \|f\|_p^p \int_{y-1}^{y+1} \exp(p\tau|v|) dv$$
$$\leqq C^p \|f\|_p^p \exp(p\tau|y|),$$

where $C = \pi^{-1/p}\exp\tau$. Thus we have the growth estimate

$$|f(z)| \leqq C\|f\|_p \exp(\tau|y|), \tag{3.4}$$

which implies (3.3). It is clear from (3.4) that each evaluation functional λ_z is continuous on $E(p,\tau)$, so $\{\lambda_z : z \text{ complex}\}$ is a separating family of continuous linear functionals on $E(p,\tau)$.

Another consequence of (3.4) is that the topology of $E(p,\tau)$ is stronger than the topology of uniform convergence on compact subsets of the plane. Thus if (f_n) is a sequence of functions in $E(p,\tau)$ which converges in the L^p norm to a function f, then (f_n) is Cauchy uniformly on compact subsets of the plane, so it converges uniformly on compact subsets to an entire function g. Since (f_n) has a subsequence which converges to f pointwise a.e., it follows that $f = g$ a.e., hence $E(p,\tau)$ may be regarded as a closed subspace of $L^p(-\infty,\infty)$. Thus $E(p,\tau)$ is complete. ///

THEOREM 2. *If* $0 < p < 1$, *then* $E(p,\tau)$ *has modulus of concavity* p, *and contains proper, closed, weakly dense subspaces.*

Proof. Let $x_n = n\pi\beta$ $(n = 0,1,2,\cdots)$, where $\tau\beta$ is an integer which exceeds $1/p$. We will show that the map T defined by

$$Tf = (f(x_n) : n \geqq 0)$$

is a continuous linear transformation of $E(p,\tau)$ onto l^p. In view of Theorem 1, the proof will then be complete. Suppose f belongs to $E(p,\tau)$. Since $|f|^p$ is subharmonic, integration over the disc $D_n = \{z : |z - x_n| > \pi\beta/2\}$ yields

$$\pi|f(x_n)|^p \leqq 4(\pi\beta)^{-2} \int\int_{D_n} |f(x)|^p dxdy$$

$$\leqq 4(\pi\beta)^{-2} \int_{-\pi\beta/2}^{\pi\beta/2} \int_{x_n-\pi\beta/2}^{x_n+\pi\beta/2} |f(x+iy)|^p dxdy.$$

Since the intervals $(x_n - \pi\beta/2, x_n + \pi\beta/2)$ are disjoint, both sides of this inequality can be summed on n to yield

$$\pi \sum_{n=0}^{\infty} |f(x_n)|^p \leqq 4(\pi\beta)^{-2} \int_{-\pi\beta/2}^{\pi\beta/2} \int_{-\infty}^{\infty} |f(x+iy)|^p dxdy.$$

Now apply Theorem A to the inner integral on the right to get

$$\sum_{n=0}^{\infty} |f(x_n)|^p \leqq C\|f\|_p^p,$$

where C is independent of f (cf. [1, p. 101]). Thus T is a continuous linear map taking $E(p,\tau)$ into l^p. We show that T is onto. Suppose (ξ_n) belongs to l^p. Let

$$g_n(z) = \left[\frac{\sin \beta^{-1}(z-x_n)}{\beta^{-1}(z-x_n)}\right]^{\tau\beta}$$

$(n = 0,1,2,\cdots)$. We will show that $g_n \in E(p,\tau)$ $(n = 0,1,2,\cdots)$ and that the series

$$\sum_{n=0}^{\infty} \xi_n g_n(z) \tag{3.6}$$

converges in $E(p,\tau)$ to a function f in $E(p,\tau)$. Since $g_n(x_m) = 0$ when $n \neq m$ and $= 1$ when $n = m$, it will then follow from the continuity of T that $Tf = (\xi_n)$, which will complete the proof.

Since $\tau\beta$ is an integer, it is clear that each g_n is an entire function of exponential type τ. Let $m = \max\{|z^{-1}\sin z| : |z| < 1\}$. Then for $n = 0,1,2,\cdots$,

$$\|g_n\|_p^p = \int_{-\infty}^{\infty} \left| \frac{\sin \beta^{-1}(x-x_n)}{\beta^{-1}(x-x_n)} \right|^{\tau\beta p} dx$$

$$= \int_{-\infty}^{\infty} \left| \frac{\sin \beta^{-1}x}{\beta^{-1}x} \right|^{\tau\beta p} dx$$

$$\leqq 2m^{\tau\beta p} + 2\beta^{\tau\beta p} \int_1^{\infty} x^{-\tau\beta p} dx$$

Since $\tau\beta p > 1$, the last integral converges, so there exists $C > 0$ such that $\|g_n\|_p^p < C$ $(n = 0,1,2,\cdots)$. Thus $g_n \in E(p,\tau)$ for each n, and it follows from the uniform bound on the norms that the series (3.6) converges in $E(p,\tau)$. ///

4. The spaces $A^p(0 < p < 1)$.

In this section we show that the spaces $A^p\,(0 < p < 1)$ of analytic functions in $L^p(|z| < 1)$ have modulus of concavity p, and contain proper, closed, weakly dense subspaces.

If f belongs to A^p, it follows from the fact that $|f|^p$ is subharmonic in the open unit disc that whenever $|z| < 1$,

$$\pi(1-|z|)^2|f(z)|^p \leqq \iint_{|w-z|<1-|z|} |f(w)|^p\,du\,dv \leqq \|f\|.$$

Thus the point evaluations $\{\lambda_z\colon |z| < 1\}$ form a separating family of continuous linear functionals, and the topology on A^p is stronger than the topology of uniform convergence on compact subsets of the unit disc. It follows as in the previous section that A^p is complete.

We will require two lemmas, whose proofs will not be given here.

LEMMA A [8, Theorem 2]. *Suppose* $0 < |z_0| < |z_1| < \cdots \to 1$, *and for some* $0 < C < 1$,

$$1-|z_{n+1}| \leqq C(1-|z_n|) \qquad (n = 0,1,2,\cdots). \tag{4.1}$$

Then there exists $\delta > 0$ such that

$$\prod_{0\leqq j<\infty,\, j\neq n} \left| \frac{z_n - z_j}{1 - z_n\bar{z}_j} \right| > \delta \tag{4.2}$$

for $n = 0,1,2,\cdots$.

LEMMA B [7, pp. 93–96]. *If* $\alpha > 1$, *then*

$$\int_0^{2\pi} |1 - re^{i\theta}|^{-\alpha}d\theta = 0((1-r)^{1-\alpha})$$

as $r \to 1-$.

THEOREM 3. *If* $0 < p < 1$, *then* A^p *has modulus of concavity* p, *and contains proper, closed, weakly dense subspaces.*

Proof. Let (z_n) be a sequence of complex numbers satisfying the hypotheses of Lemma A. We claim that the mapping T defined by

$$Tf = (f(z_n)(1 - |z_n|)^{2/p} : n \geqq 0)$$

is a continuous linear transformation taking A^p onto l^p. For convenience, write $r_n = |z_n|$. If f is in A^p, and D_n is the open disc of radius $(r_{n+1} - r_n)/2$ centered at z_n, then we have from the subharmonicity of $|f|^p$ that

$$\pi(r_{n+1} - r_n)^2 |f(z_n)|^p \leqq 4 \iint_{D_n} |f(w)|^p \, dudv. \tag{4.3}$$

From (4.1) it follows that $r_{n+1} - r_n \geqq (1 - C)(1 - r_n)$. Substitute this inequality into (4.3), and sum both sides of the resulting inequality on n. Since the discs D_n are disjoint $(n = 0,1,2,\cdots)$, it follows that

$$\sum_{n=0}^{\infty} |f(z_n)|^p (1 - |z_n|)^2 \leqq 4 \| f \| / \pi (1 - C)^2 .$$

T is therefore a continuous linear mapping of A^p into l^p.

We show that T is onto. If (ξ_n) is in l^p, form the series

$$\sum_{n=0}^{\infty} \xi_n b_n(z) b_n(z_n)^{-1} g_n(z), \tag{4.4}$$

where

$$b_n(z) = \prod_{0 \leqq j < \infty, j \neq n} \frac{\bar{z}_j}{|z_j|} \frac{z_j - z}{1 - \bar{z}_j z},$$

and

$$g_n(z) = (1 - \bar{z}_n z)^{-\beta} (1 - |z_n|^2)^{\beta - 2/p},$$

where $\beta > 2/p$. We claim that the series (4.4) converges in A^p. Once this has been established, if f denotes the limit function, then $f \in A^p$, and it follows from the continuity of T that $Tf = (\xi_n)$.

Let f_n denote the nth partial sum of (4.4) $(n = 0,1,2,\cdots)$. It follows from Lemma A that $|b_n(z_n)| > \delta > 0$, and from standard function theory ([11, page 302], for example) that $|b_n(z)| < 1$ whenever $|z| < 1$. Thus whenever $m > n$, we have

$$\| f_n - f_m \| \leqq \delta^{-p} \sum_{j=n+1}^{m} |\xi_j|^p \| g_j \| .$$

A straightforward calculation employing Lemma B shows that $\| g_j \| < M$ $(j = 0, 1, 2, \cdots)$ for some $M > 0$. Thus (f_n) is a Cauchy sequence, and the series (4.4) converges in A^p, proving our assertion about T. It follows as before from Theorem 1 that A^p has PCWD subspaces and modulus of concavity p. ///

The discussion of A^p carries over to the more general classes $A(\alpha)$ $(\alpha > -1)$ of functions f analytic in the unit disc with

$$\| f \| = \iint_{|z| \leqq 1} |f(z)|^p (1 - |z|)^\alpha dx dy < \infty$$

The results are identical, and the proofs are essentially the same. The main difference is that in deriving the necessary inequalities, instead of simply integrating over discs, one must integrate over appropriate annuli with respect to a Poisson kernel. In addition, certain mixed norm spaces of functions analytic in the disc have been treated by this method [13].

Theorem 1 also applies to the spaces H^p $(0 < p < 1)$. It can be shown that the result in [12], which shows that the mapping

$$f \to (f(z_n)(1 - z_n)^{1/p})$$

takes H^p $(1 \leqq p < \infty)$ continuously onto l^p if and only if (z_n) satsfies (4.2), remains true when $0 < p < 1$ (see [3, ch. 9]). Thus H^p $(0 < p < 1)$ contains PCWD subspaces, and has modulus of concavity p.

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